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Estimating Investors' Behavior and Errors in Probabilistic Forecasts by the Kolmogorov Entropy and Noise Colors of Non-Hyperbolic Attractors

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ABSTRACT

This paper investigates the impact of the Kolmogorov-Sinai entropy on both the accuracy of probabilistic forecasts and the sluggishness of economic growth. It first posits the Gaussian process Z_t (indexed by the Hurst exponent H) as the output of a reflexive dynamic input/output system whose attractor is non-hyperbolic. It next indexes families of attractors by the Hausdorff measure (D_0) and assesses the uncertainty level plaguing probabilistic forecast in each family. The D_0 signature of attractors is next applied to the S&P-500 Index The result allows the construction of the dynamic history of the index and establishes robust links between the Hausdorff dimension, investors' behavior, and economic growth

Keywords: *Stochastic processes, Hausdorff dimension, forecasts, entrupy, chaotic attractors, investors' behavior, economic growth.*

1. INTRODUCTION

Economists and architects of financial theories have spent a considerable time studying random stochastic fields, and sequences of random variables evolving in time, hoping to find one that would mimic the evolution of prices and or returns of financial variables. A large portion of the ensuing literature is devoted to processes such as α-stable Levy, self-similar, random walk, brownian motion with drift, other constructed Wiener processes, etc. with particular emphasis on understanding "volatility clustering", which in turn would improve on prediction. This paper is intended to show, among other things, the difficulty of reaching these objectives if one focusses solely on the properties of statistical distributions of processes rather than on their driving mechanisms.

To be more specific, the experts wanted to know whether or not the property known as "long-term dependence" (LTD) exists in financial returns. LTD, another name for volatility clustering according to Mandelbrot [1], hinges either on specifications such as ARCH and GARCH (1 1), etc. or on the behavior of autocorrelation functions at large lags. If the autocorrelation function is observed to decay at a geometric rate, they conclude that short-term dependence (STD) exists; whereas a decay rate that is as slow as a power law decay is assumed to indicate long-term dependence (LTD).

There are, however, a few drawbacks associated with the latter procedure. The nonstationarity of returns may generate spurious results that may be interpreted as LTD. When LTD and heavy tails occur together, autocorrelation functions may fail to be viable estimators. And there exist many decay rates between geometric and power law. Hence, results from either autocorrelation functions or the GARCH specification do not convince, because they explain neither "market memory" nor the "dependence" of future volatility on present volatility. There are also two important omissions in the autocorrelation approach that stand to cloud the picture even further. The first is that financial time series are outputs of some "complex" dynamic systems. Even though position, behavior, and the future course of these systems are fraught with uncertainty (see [2], [3], [4], [5], among others), there are, however, no substitutes for them at this juncture. The second omission relates to the total neglect of STD, which is full of pertinent information about the state of dynamic systems, as other sciences such as seismology, acoustics, hydrology, etc. have already discovered. If markets are dynamic constructs whose outputs (in terms of time series) vary in space and time, then variations of this nature reinforce the belief that modern markets may indeed have attractors that are either "strange" or "complex" or even "chaotic". And to shed light on the behavior of market attractors, there is no viable alternative but dynamic analyses.

This paper will steer in that direction. The second part will assume that market prices are observed outputs of complex dynamic systems that are best depicted by the so-called "fractional Brownian motion" (fBm), originally proposed by Mandelbrot and van Ness [6], and augmented to "Mixed fractional Brownian motion" (MfBm) by Zili [7], Thale [8], Dominique and Rivera [9], among others; in an MfBm process, for example, it is easier to pinpoint the shrtcomings of autocorrelation functions..Part III will first briefly discuss the theoretical background and our research method. Then it will use a family of Iterated Function Systems (IFS) to index, and to color-code families of attractors by the Hausdorff measure as a way of bringing to the fore the extent of the Kolmogorov-Sinai entropy, assumed to be the main source of the pervasive uncertainty associated with the motion of prices. The findings of Part III will be used in Part IV to examine the S&P-500 Index (taken to be a combination of Gaussian processes) in an attempt to determine whether attractors' noise colors reflect investors' behavior as an important determinant of economic growth.

2. THE MIXED FRACTIONAL BROWNIAN MOTION

Consider first the Mandelbrot-van Ness' specification:

$$X_t^H = \{X^H(t, \omega), t \in \Re, \omega \in \Omega\}. \qquad (1)$$

It is a real-valued intersection of self-similar and Gaussian processes defined on (Ω, Σ, P}, where each is indexed by $H_i \in (0, 1)$, satisfying $E(X^{H_i}(t, \omega)) = 0$, $\forall t \in \Re$. Here, E denotes the

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expectation with respect to the probability law P for X^{Hi}; (Ω, Σ) is a measurable space; H is the Hurst [10] exponent, and X_t^{Hi} are considered non-observable inputs into the observable output Z_t given by:

$Z_t = \sum_i b_i (X_t^{Hi})$, where $b_i \varepsilon \Re$, i ε n, H_i ε n, $\forall H_i \varepsilon (0, 1)$. (2)

Z_t is a linear combination of quasi self-similar Gaussian processes or a superposition of n independent input streams (X_t^{Hi}), each with its own H. Z_t has stationary but correlated increments, and it is invariant under a whole family of transformations. As in input storage and teletraffic, X_t^{Hi} is assumed to arrive into Z_t as "cars" (short-term expectations) or as "trains" (long-term expectations).

Z_t is conventionally assumed to be completely characterized by its zero mean and its covariance function, given by:

$$\text{Cov}(Z_t, Z_s) = R(t, s) = 2^{-1}\sum_i^n (b_i)^2 [\, t^{2Hi} + s^{2Hi} - |t - s|^{2Hi}, \quad \forall\, t, s \,\varepsilon\, \Re,\ i\ \varepsilon\ n. \qquad (3)$$

It is shown in Appendix A1 that $d^2(\text{Cov}(Z(t,s; H^i))$ is neither smooth nor monotone due to scaling variations. As a consequence, the autocorrelation function, denoted C (τ), where τ is the lag, being:

$$C(\tau) = C(Z_t, Z_{t+\tau}) = [L(t, h)\,(1 / (\tau^{(1-2c)}))] / L(t) = 1/(\tau^{(1-2c)}), \qquad (4)$$

where $L(t, h) / L(t) \to 1$ as t, $\tau \to \infty$, $0 < c < ½$, and $H_i > ½$; that decay rate is observable in the *persistence* region. Whereas in the *anti-persistence* segment of H, the decay rate is given by:

$$C(\tau) \le A\, c^{\tau} \text{ for } A > 0,\ c\ \varepsilon\ (0, 1) \text{ and } H_i < ½. \qquad (4')$$

Equations (4) and (4') imply that for $H_i > ½$, C (τ) decays as *a power law* of the lag τ as t moves forward. But for $H_i < ½$, the autocorrelation function decays at a *geometric* rate of the lag. Appendix A1 shows that $H_i = ½$ is a (non-Morse) degenarate critical point of (3), the location of a *cusp*. Put differently, for reasons that will be explicited later, (3) is in reality not strictly concave at $H_i < ½$ nor strictly convex at $H_i > ½$. It may also be noted that the usual assertion to the effect that a Brownian motion is recovered at $H_i = ½$ is not supported by (A1.2), (A1.3) or by (4) and (4'). This is not surprising because Brownian motions are idealized and mathematically convenient intersections of *self-similar* and $\propto$ - *stable Levy* processes.

These results provide an explanation as to: 1) why the probability law in (A1.1) offers so little guidance to forecasters; 2) why the jump in decay rate in (4) and (4'), and 3) why it is so difficult to come up with a convincing explanation for the presence of LTD, or volatility clustering. In fact, volatility clustering only refers to the first part of the probability law in (4), while the likelihood of high frequencies in the H < ½ region is all but ignored.

It is more promissing, it seems, to focus on the signal to noise ratios, or on upper and lower bounds in information thrown-off by dynamical systems. Recalling also that statistical mechanics allows for macroscopic predictions based on micro-properties of systems mediated by entropy, which may be interpreted as a description of how information changes as systems evolve from their initial states [11], [12]. And such changes are measured by the so-called Kolmogorov-Sinai entropy which, according to the Persin's Theorem [13], may easily be evaluated by the Lyapunov characteristic exponents (LCE) or λ Mainly for tractability, therefore, the next section briefly reviews the basic notions needed for an evaluation of information in dynamic systems.

3.THEORETICAL BACKGROUND, METHOD, AND NOISE COLORS OF ATTRACTING SETS

Attractors that exhibit chaotic behavior are generally divided into three types, namely, hyperbolic, quasi-hyperbolic and nonhyperbolic; but there is little difference between the first two. Quasi-hyperbolic attractors (like the Lawrence's type) are so close to hyperbolic ones, differing only in the absence of tengencies, that we will consider only two brorad categories, that is, hyperbolic and nonhyperbolic. For the present purpose, however, suffice it to focus on one-dimensional non invertible maps. But to stress some differences in the route to chaos and to define certain terms needed for the rest of the discussion. we confine the basic characteristics of multi-dimensional hyperbolic attractors to Appendix A2.

We will not pursue the argument described in Appendix A2 for two main reasons. First, there is no clear consensus on the definition of chaos. For our point of view, Smale's horseshoe ends up in a 'strange ' attractor, which we do not consider chaotic. Instead we will adopt the definition of Echmann and Ruelle [14], which associates chaos with the presence of sensitive dependence on initial conditions (SDIC). And second, hyperbolic attractors are essentially mathematical constructs. Attractors that are observed in real-world applications are nonhyperbolic. In those, global manifolds M^s (.) and M^u (.) intersect at zero angle (Appemdix A2) and, differently from hyperbolic attractors, some experts (see below) have argued that in general, they are more sensitive to noise over their chaotic intervals. Just this latter feature, if true, would make them worthy of interest in economics where data are always noisy due to imperfect information.

This paper considers a 1-D non-invertible non-linear map in the form of $x_{n+1} = f(x_n; K)$, where K is the system's parameter. It will be shown that the qualitative behavior of the solution set of f changes as the C^1 vector field passes through points of the bifurcation set at which f is not structurally stable. More specifically, we will assume throughout that $f = C^1 (K \times I_l)$, where K is an open set in $\Re$ and $I_l \subset \Re$ is an interval. As K is increased from below, various qualitative behaviors are observed. There is a K-value, K_∞, at which a strange attractor appears mainly due to unequal schrinking and stretching of the unit interval as orbits move to accumulation points. When the K is increased further a phenomenon described by Kudo *et al.* [15] appears. That is, low period orbits becomes transversally unstable and both high and low-dimensional chaos are observed (*vide infra*). These attractors will be differentiated and indexed by their Hausdorff measures.

This method will show that chaotic behavior does in fact appear in non-invertible 1-D maps due to singularities and non-invertibility associated with the mixing process. Thus, a 1-D non-invertible map is a simple tool for understanding the topological structure of flows in dynamical systems.

As already noted above, each segment of Z_t is the output of an attractor which may be strange or chaotic. As Z_t varies over time (and space) (see [16], [17] and an interesting paper by Peciado and Morris, presented at the World Congress on engineering and computer science held in San Francisco in 2008), it is essential therefore that the whole family of attractors be examined before Z_t can be completely characterized.

Appendix A2 is a reminder that the route to chaos may passes through Smale's horseshoe. But in 1-D non-invertibe maps, the route passes through the cascade of period doubling. We will then show that the quadratic map shown in (6) below generates the following well-known types of attractors as the bifurcation parameter varies. That is, :

i) Stable hyperbolic fixed-point attractors; eigenvalues are less than zero at the equilibrium point z_{*};

ii) Period-doubling cascade attractors of periodic orbits $\Gamma$of the form Γ-2^k, k = 0, 1, 2, 3, 4, …

iii) Fractal attractors are nonhyperbolic and they occur at the upper limit of the period-doubling cascade. These resemble Cantor-point sets and they contain a countable set of periodic orbits of large periods, an uncountable set of aperiodic orbits, and a danse orbit;

iv) High-dimensional chaotic hyperbolic attractors wirh, say, $q \varepsilon \Re_+$ degrees of freedom, but they are not sensitively dependent on initial conditions (SDIC);

v) Low-dimensional chaotic hyperbolic attractors, i. e., with p < q degrees of freedom, and SDIC.

In multi-dimensional diffeomorphisms (Appendix A2), high dimensional chaotic attractors are often called "complex". The sum of their positive Lyapunov characteristic exponents (λ) exceeds the sum of their negative λs. Clearly their number of *effective* degrees of freedom is lower than the dimension of their embedding space but higher (hence higher Hausdorff dimension (D_0)) than that of "chaotic" attractors. On the other hand, attractors that we term "chaotic", or low dimensional chaotic, do have SDIC, and still a lower number of effective degrees of freedom than complex attractors. Complex attractors' ability to reduce their effective degrees of freedom may be due to the concept of self-organization criticality whereby they reduce their own entropy by discharging chunk of it to an external reservoir to satisfy the Second Law of thermodynamics.

In nonhyperbolic attractors, non-hyperbolicity derives from intersection tangencies of M^s and M^u, and "Unstable Direction Variability" (UDV), which is one of the fingerprints of complexity. UDV arises in nonhyperbolic systems when unstable periodic orbits with distinct number of unstable directions are densely mixed; that is, each unstable periodic orbit has a different number of expanding directions.

In 1-D map such as (5), complicated behavior arises from the battle between the action of schrinking and stretching on the unit interval. In other words, during period-doubling, contraction overshadows expansion up to K_∞. When the parameter K is increased beyond K_∞, the opposite occurs; that is, expansion overcomes contraction to produce complexity or high-dimensional chaos. In the period-three interval, contraction once again overcomes expansion. And past that interval, expansion returns but not to the same extent as in high-dimensional chaos. In all, contraction seems to produce stability, while expansion seems to be associated with unstability .

Before concluding this section, let us recall that the Hausdorff dimension of an attractor is a non-probabilistic measure of how orbits fill up the space available to them. Estimating the D_0 of segments of Z_t is therefore an easy way of identifying complex and chaotic attractors, as will be shown in the next section. But for now, let us reemphasize two points that will guide the reader for the rest of the discussion. That is, the hallmark of chaotic attractors is SDIC which may be quantified in terms of λs which, in turn, measures the rate of exponential divergence of nearby trajectories. The second is the concept of the Kolmogorov entropy quantifiable by the sum of positive λ^+_s, which measures the rate of information not available in dynamic systems. The next section will use iterated function systems (IFS) to first classify nonhyperbolic attractors by broad sub-classes, and next will attempt to estimate the level of entropy in each sub-class.

3.1 Indexing Attractors with the Hausdorff Dimension

The easiest way to separate non-hyperbolic attractors by broad sub-classes and index them either by the Hurst exponent or the Hausdorff measure is to consider a family of Iterated Function Systems (F) (see [18] and references therein). That is,

$$F = \{f_i\},\ f_i: (0, 1) \to (0, 1),\ i\ \varepsilon\ n, \text{ where}$$

$$f_i (z_t) = z_{t+1} = K z_t – K h z_t^{\eta + 1}, \qquad (5)$$

Equation (5) is quite general. Its mean (z^{--}), control factor (K), equilibrium (z*) and stability condition are given by:

$z^{--}= \{1/[h (\eta + 1)]^{1/\eta} \}$, $K = 1/[z^{--} (1 – h z^{--\eta}]$, $z_* = [(K - 1) / h K]$ and $df_i/ dz_t = K [1 – h z^\eta (\eta + 1)]\ \varepsilon\ [-1, 1]$.

However if h =1 and η = 1, (5) is reduced to a modified version of Jean-Francois Verhult's growth equation , a. k.a, the logistic parabola, which is non-hyperbolic. In that form, solutions exist for $1\leq K \leq 4$, but solutions for $1\leq K \leq 3$ yield fixed-point attractors, while those between $3 < K \leq 4$ generate period-doubling, complex, and chaotic attractors. Another important feature to note is that every f_i between $3 < K \geq 4$ may be indexed either by H ε (0 ,1) or by D_0 ε (1, 2), starting with the highest member, denoted f^{max} for which H = 0 or $D_0 = 2$. Thus, $K = 4 - H = 2 + D_0$ describes any f in the range of D_0 and indexes its output at the same time. Equation (5) can then be written as:

$$f(z) = (2 + D_0)\, z_t\, (1 - z_t), \text{ for } h = 1, \eta = 1 \quad (6)$$

The iteration process identifies the attractors of interest, beginning with the first bifurcation given by:

$$f(f(z)) = K\{ K z (1- z) [1–K z (1– z)]\} = z; \quad (7)$$

and

$$- K^3 z^{*4} + 2 K^3 z^{*3} – (K^3 + K^2)\, z^{*2} + K^2 z = z^* \quad (8)$$

In (7) z = ½ is the first superstable orbit encountered ($df_i /dz = 0$) and it happens to be a member of the list of equilibria of all hiher iterates. The values of K for the first bifurcation are $K_1 = 2$, and $K_2 = 1+ (5)^{1/2}$. Equation (8) can be solved for the stable equilibrium values of $z_{1*} = ½$ for $K_1 =2$ and $z_{2*} = 0.809$ for $K_2 = 3.23$. The nest iterates are f (f (f (z))) = z, f (f (f (f (z)))) = z, etc. The result is such that every f_i has a D_0 signature distinguishing it from another, and each describes a prototypical attractor.

We use the Wavelet Multi-resolution Benoit_{TM} of Trusoft International to compute D_0 in the range of $3 < K \geq 4$. The result given in Figure 1 and Table 1 below allows for 5 sub-classes of attractors according to their characteristics. The pertinent conclusions that can be drwn are as follows:

In one-hump maps such as (6), centered at z = ½, scale invariance is theoretivally be broken (not at 3 but) at K = 3.23, which is a root of (7). However, as a first surprise, scale invariance is experimentally broken well after a phase shift located before the second bifurcation occurring at K = 3.49. Incidentally, the fact that scale invariance is broken beyond the first bifurcation explains the black band observed at the beginning of the phase of period-doubling (see Figure 2). The second surprise is that the lower bound of the Hausdorff measure is at $D_0 = 1.08$ rather than 1.0. The third is that in chaotic regimes, whether high or low-dimensional, equilibria are hyperbolic and the system is sensitive to noise.

In *period-doubling* attractors, stable equilibria become unstable just before bifurcating. However, LCEs remain negative; a similar remark applies to the Li and York's [19] window at 3.82…≤ K ≤ 3.84…

In *fractal or strange* attractors, LCEs are zero as orbits are either periodic or aperiodic, but no SDIC is observed. Consequently, there is no chaotic motion in the sense of Eckmann and Ruelle [14].

Attractors that exhibit *complex* motion are also high-dimensional chaotic. Their D_0 is higher than those of other types, indicating thereby the *highest level of Kolmogorov-Sinai entropy*. Attractors in that sub-class are able to decrease their entropy by dissipation. That is, they simply organize motion around fewer effective degrees of freedom by dischharging entropy into an esternal reservoir. However, until they can self-organize themselves, the area of their attracting set visited by orbits is the largest by definition. Their average D_0 is approximately 1.7.

The sub-class of low-dimensional *chaotic* attractors has a lower D_0 than the above category; their average D_0 is about 1.3. A good example of that type is the Henon map even though the latter is 2-ditmensional and posesses a number of features that distinguish them from (6).

It is tempting at this juncture to confront these results with computed values available in the literature. It is found that complex attractors have a larger $D_{0,}$ averaging 1.7, and the *area* of the attracting set visited by orbits is also larger than *that* of chaotic attractors. For example, the complex Ikeda map:

$$x_{n+1} = 1 + 0.9\, x_n \exp\{0.4i – [6i / (1 + |x_n|^2)]\}, \;\; x_{n,}\, x_{n+1}\, ε\, M,$$

belongs to a family of high dimensional chaotic attractors. Its average D_0 is 1.7 and the area of the attracting set visited by orbits is larger than that of the the Henon map:

$$x_{t+1} = 1.0 + y_t – a\, x_t \quad \text{and} \quad y_{t+1} = b\, x_t\, ;$$

for a = 1.4, and b = 0.3, its $D_0 = 1.3$. Moreover, the subset of the attracting set B visited by orbits is much smaller than that of the Ikeda attractor. In (6) at K = 4, one enters into a regime of complete chaos with a $D_0 = 1.42$. Equally in the Henon attractor, at a > 1.55, all orbits similarly escape to infinity.

Finally, it is hypothesized that that strange attractors should appear as a Cantor point-set. Grassberger [20] has carried out theoretical and analytical computations on a map such as (6) in 1-D and has found a $D_0 = 0.538…$ at aperiodicity, close to the Cantor set whose $D_0 = 0.6309$. In this study, the value found in 1-D is $0.58 < D_0 < 0.59$.

3.2 Assessing the Entropy Level

The laws of physics do not permit the destruction of information. Then, the Kolmogorov entropy must be interpreted as a a measure of information not available. Figure 2 (Plate a) below is a schetch of the findings of Table 1. The Kolmogorov entropy, being the sum of positive LCEs of an attractor, is presented in Plate (a) of Figure 2 as the depths of the valleys for each sub-class. As it can then be seen, a significant amount of entropy (or information (I) not available) is present in complex attractors. *This then explains why probabilistic predictions would be inadvisable for time series whose D_0 falls in the ranges of 1.6 to 2.0 and 1.2 and*

1.42. Whereas in the black regions of Figure 2, representing linearity, predictions should be a trivial matter.

As this procedure is an objective measure of market' expectation, it may be seen as substitute for that of Chicago Board Options Exchange Market Volatility Index (VIX).

4. INFORMATION, INVESTORS' BEHAVIOR, AND ECONOMIC GROWTH

As shown in Figure 2, black noise output depicts fixed-point attractors of monofractal processes. Economists call them "perfectly competitive" processes in which participants are assumed to be small but have all the market information available. The main drawback in such idealized set-ups though is the total absence of economic growth. This seems to suggest that growth is generated by imperfect competition, but in that other setting, information sets of participants are necessarily incomplete, which is an additional sourse of noise further constraining viable information. This section will show that the more incomplete are information sets of participants (hence higher entropy) the lower is the real rate of economic growth.

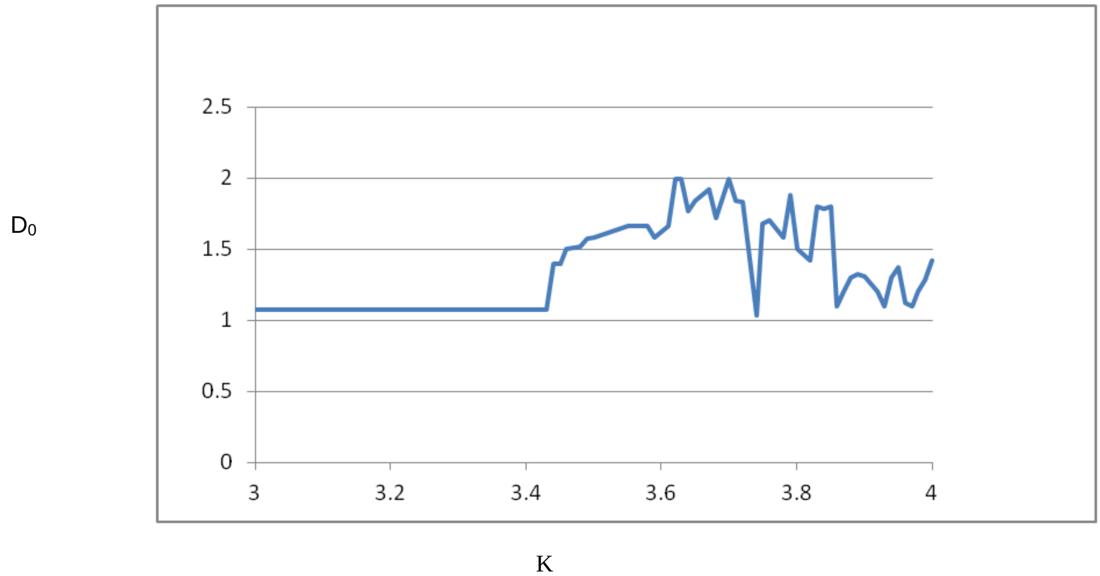

Figure 1: D_0 vs. K. Over the interval 3.0 < K < 3.44, the process is a persistent monofractal. Over the in terval 3.45 < K < 3.84 it is an anti-persistent multifractal. The interval 3.82…. ≤ K ≤ 3.84 …is the Li & Yorke's period-3 window. The interval 3.84 < K < 4.0 is persistent multifractal and chaotic.

Table 1: Salient characteristics of sub-classes of attractors indexed by the Hausdorff dimension D_0.

Type of attractors	Range of D_0	Concentra-tion	Power spectral density (β)	LCEs (λ)	Noise colors	Remark
Fixed-point	$1.0 < D_0 < 1.2$	---	$2.6 < \beta < 3.0$	< 0	Black	Experimentally, lower bound of D_0 is 1.08
Period-doubl-ing	$1.41 < D_0 < 1.59$	---	$1.8 < \beta < 2.18$	< 0	Dark pink	Black band at $1.42 < D_0 < 1.47$
Fractal	$D_0 \approx 1.59$	---	$\beta \approx 1.82$	0	Dark pink	≈ Cantor-point set; no SDIC

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Complex	$1.59 < D_0 < 2.$	$D_0 \approx 1.7$	$1.0 < \beta < 1.8$	> 0	Pink	No SDIC; high dimensional chaos, but there exist a few stable orbits
Chaotic	$1.2 < D_0 < 1.42$	$D_0 = 1.3$	$2.16 < \beta < 2.6$	> 0	Dark grey	SDIC; low-dimensional chaos, but there exixt a few stable orbits

4.1 The Correlation of Economic growth and Investors' Behavior

This section uses the Grand Microsoft Excel data set of the S&P-500 Index, sampled daily from January 3rd 1950 to February 28th 2011. The index was first detrended using logarithmic differences and divided into 12 segments of various lengths and according to quasi self-similar scales. Each segment was next filtered for white noise. For each segment, the Hausdorff dimension (D_0) was computed using the Wavelet Multi-resolution Benoit$_{TM}$ of Trusoft International. Next, the D_0 of each segment was matched with the average real rate of growth of the U. S. economy over the same period. The reader should be made aware, however, that this a coarse comparison since the series of real

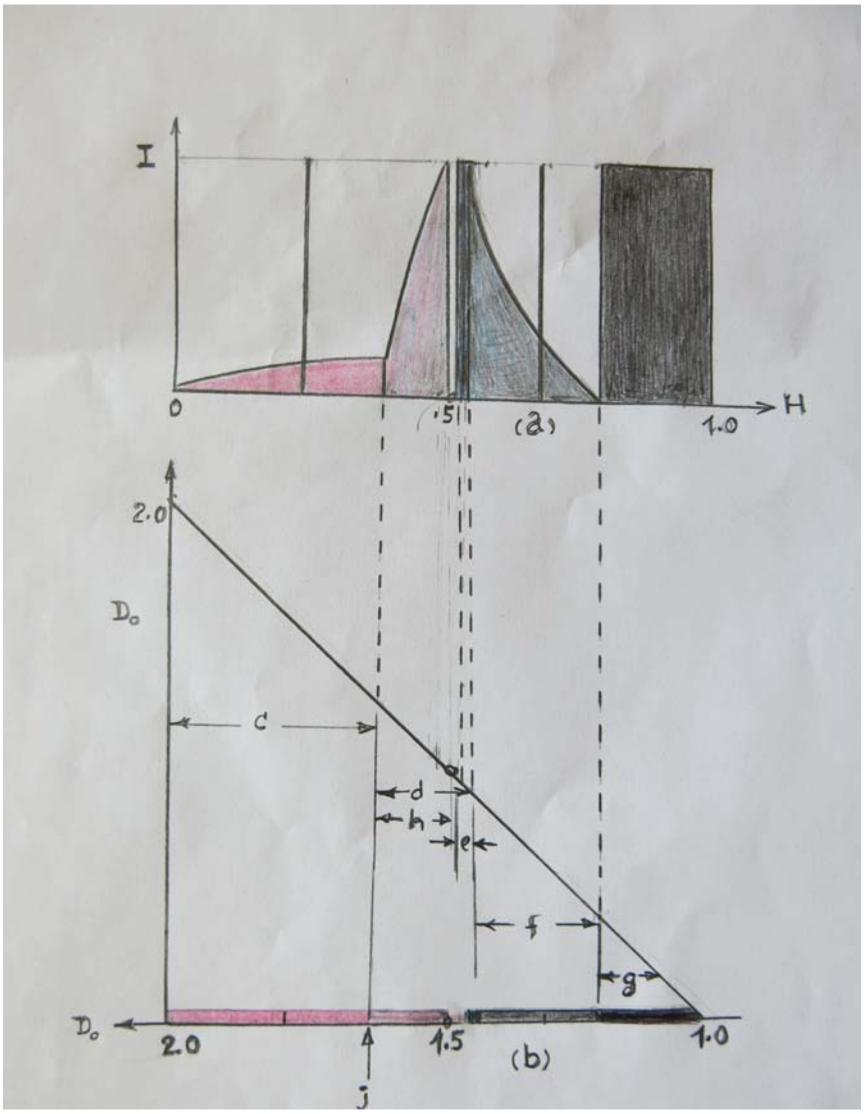

Figure 2 : Families of attractors indexed by the Hurst exponent (H) and the Hausdorff dimension (D_0). Plate (a): Information level vs. H. The depth of the valleys (white space) indicates the extent of infornation not available due to the

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Kolmogorov entropy. Plate (b): Distance c is pink; distance d is dark pink and black (e); distance f is dark grey; distance g is black, point j indicates the location of strange attractors.

Table 2: Noise Color vs the average yearly rate of growth of real US GDP, 1950-2011. (1) Real US GDP in 2005 dollars from US Bureau of Economic Analysis, retrived from www.BEA.gov on March 26th 2013. (2) Exception to the color-coded rule.

Period	D_0	Average Yearly Rate of Growth of Real GDP in Percent (1)	Noise Color
1950- 58	1.52	3.56	Dark Pink
1958-61	≈ 1.41	4.10	Dark Pink
1961-72	1.47	5.69	Black
1983-87	1.44	4.80	Black
1988-92	1.47	3.32	Black
1992-97	≈ 1.54	3.78	Dark Pink
1998-02	1.39	3.08	Dark Grey
2003-07	1.89	2.31	Pink
2007-08	≈ 1.72	0.99	Pink
2009-11	≈ 1.86	1.06	Pink
1972-80 (2)	≈ 1.78	3.20	Pink

economic growth rates was sampled at yearly intervals. Despite this lack of sampling concordance between the series, however, the assertion of a correlation between investors' behavior and economic growth is borne out from at three points of view.

In teletraffic, attractors' characteristics vary with inputs' arrivals. Similarly in this study, markets characteristics vary with participants' behavior. Put differently, investors' confidence seems to hinge on information availability. The results are given in Table 2 below.

Observe first that none of the segments of the index falls within the interval $1.0 < D_0 > 1.2$. This is interpreted to mean that over the sampling period the U S economy could not have been characterized by perfect competition. However, as already indicated above, scale symmetry is not broken at the expected theoretical value. Hence, when the economy was either within the interval $1.42 < D_0 < 1.47$ (black band) or within the interval $1.47 < D_0 < 1.6$ (dark pink) or the remaining of the period-doubling region, the average rate of real growth was the highest on record. Thus, the conclusion to the effect that growth occurs when investors can rely on market information seems compelling. Further, the period-doubling region being composed of period-2 cycles, it also means that investors had no trouble "riding" these cycles, as the entropy level is low. The only exception to that rule is during the period 1972-80, when the economy was in a period of heavy inflation, i. e. in one case on twelve.

Figure 2, shows families of attractors indexed by the Hurst exponent (H) and the Hausdorff dimension (D_0). Plate (a) plots Information level vs. H. The depth of the valleys (white space) indicates the extent of infornation *not* available due to the Kolmogorov entropy. In Plate (b), distance c is pink; distance d is dark pink and black (e); distance f is dark grey; distance g is black, point j indicates the location of strange attractors.

The other point of interest is that the real rates of growth appeared low when the market noise was pink. In the same vein, it is interesting to note the evolution of the average rate of growth of fixed assets over the pink region of Plate (a) of Figure 2. For example, from 2004 and 2011, private fixed assets, measured in billions of 2005 dollars, grew at an average yearly rate of 0.59 percent, while that of total fixed assets was even lower at 0.20 percent. The obvious explanation that can be found for this state of affairs is that investors' long-term perspective was chattered by the extent of the Kolmogorov entropy during that period. That also may have forced them to become "short-termists" relying more and more on high frequency trading, which seems anathema to economic growth.

Finally, Table 2 shows that at no time the market was low-dimensional-chaotic in the sense of Scheinkman and Le Baron [21].

As hypothesized, the real rate of growth appears low when the noise is color-coded pink. If noise colors were to vary as K remains constant, such changes could be attributed to the phenomenon called "intermittency" of stable orbits wandering into E^u before returning to E^s or due to an incomplete intersection of E^s and E^u, or even due to a reduction of effective degrees of freedom). The fact that K underwent changes rules out intermittency. In all and to be consistent with the initial hypothesis, it seems compelling to conclude that markets are indeed *reflexive dynamic input/output constructs.*

5. CONCLUSIONS

This paper posits process Z_t as the output of a dynamic system whose attractor may be period-doubling, strange, complex or chaotic. When Z_t is modeled as an MfBm, it reveals the limitations of the conventional approaches used to explain volatility clustering and to predict future values of financial stochastic processes. This study instead explains volatility clustering as a matter of investors' confidence, and establishes the scaling histories of these processes in terms of the noise colors of their attractors.

In that context, iterated function systems were next used to index various sub-categories of non-hyperbolic attractors by their Hausdorff measure (D_0). As a result, it is found that the D_0 of period-doubling attractors lies between 1.41 and 1.59; that of strange attractors is about 1.59, while the average values of complex and chaotic attractors are 1.7 and 1.3, respectively. In addition, that approach allows the assessment of the Kolmogorov-Sinai entropy of each sub-class of attractors, which appears to be the root cause of the pervasive uncertainty surrounding probabilistic forecasts. Moreover, these results are generic to the whole family of quadratic maps.

The approach also shows that control parameter variations can bring about a transition from one type of attractor to another. Under this transition a robust nonhyperbolic attractor can be transformed into a quasi-hyperbolic or hyperbolic one, and vice versa. The important point to stress, however, is that chaotic attractors have two fundamental properties, namely complex geometric structure and exponential instability of individual trajectories.

The above results were then applied to the S&P-500 Index as a proxy for the U. S. economy. It is found that, over the period 1950-2011 examined, the D_0 of the attractor of the economy fluctuated in values between those of period-doubling and complex. Interestingly, it is also found that the yearly real rates of economic growth were significantly higher when the noise color of the attractor was either black or dark pink rather than when it was pink. As the dynamic system in question is obviously reflexive, the paper concludes that input behavior begetting changes in attractors' noise colors is therefore a major determinant of economic growth.

APPENDIX A1: INTERPRETING THE PROBABILITY LAW

The probability law in (1) is interpreted as follows: Denote a positive move at time t as Δz_t^+ and a negative move as Δz_t^-. Similarly for a positive and negative moves at t + 1 as Δz^+_{t+1} and Δz^-_{t+1}, respectively. Next assign probability p_1 to Δz^+_{t+1} and p_2 to Δz^-_{t+1}. Then:

Given $H > ½$, and (Δz^+_t), then $p_1 (\Delta z^+_{t+1}) >> p_2 (\Delta z^-_{t+1})$

$H > ½$, and (Δz_t^-) : $p_2 (\Delta z^-_{t+1}) >> p_1(\Delta z^+_{t+1})$. (A1.1)

Given $H < ½$, and (Δz_t^+), then : $p_2 (\Delta z^-_{t+1}) >> p_1 (\Delta z^+_{t+1}$

$H < ½$, and (Δz_t^-): $p_1 (\Delta z^+_{t+1}) >> p_2 (\Delta z^-_{t+1})$

Given $H = ½$: $p_1 = p_2$ and $p_1 + p_2 = 1$

In words, if H > ½ and today's move is positive, then the probability of having a positive move tomorrow (p_1) is much greater than the probability of having a negative move tomorrow (p_2). Similarly, p_2 is much greater than p_1 if todays' move is negative and H > ½, etc. However, at H = ½, $p_1 = p_2$ is equivalent to having no useful information.

Given the conventional interpretation, why then probabilistic forecasts are so wide-off their marks? To proffer an answer, consider a vector u = [t. s], where t > 0, s > 0, t > s, and the second derivatives of (3). Denote $R_{tt} = \partial [\partial R_{t,s}) / \partial t] / \partial t$ and similarly for R_{ss} and $R_{ts} = R_{st}$, etc. Then:

$$R_{tt} = 2^{-1} \sum_i (b_i)^2 (2 H_i) (2 H_i - 1) [t^{2Hi-2} - |t - s|^{2Hi-2}]$$

> 0 for $H_{i>}= ½$; 0 for $H_i = ½$; < 0 for $H_i < 1/2$ (A1.2)

$$R_{ss} = 2^{-1} \sum_i (b_i)^2 (2 H_i) (2 H_i - 1) [s^{2Hi-2} + |t - s|^{2Hi-2}]$$

> 0 for $H_i > ½$; 0 for $H_i = ½$; < 0 for $H_i < ½$.

$R_{ts} = R_{st} = 0$ for $H_i = ½$

From the Hessian matrix (Ħ) of (A1.2), we have :

i) u^T Ħ $u > 0$, i.,e positive definite for $H_i > ½$

ii) u^T Ħ $u < 0$, negative definite for $H_i < ½$.

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iii) det (Ħ) = 0 for H_i = ½, (A1.3)

where the T stands for the transpose operator.

APPENDIX A2: CHAOTIC HYPERBOLIC DIFFEO-MORPHISMS

Let **g**: $\Re^m \rightarrow \Re^m$ be a diffeomorphism of a smooth global manifold, where N (B) is the neighborhood of B, and φ_t (B) is the flow or the evolution operator telling how the state of the system changes over time. If φ_t (B) ⊂ N (.) at t ≥ 0 and φ_t (,) →B as t →∞ , then B may be a compact *hyperbolic* attracting set for **g**. Moreover, in such a dissipative system, it can easily be shown that the volume of a fiducial phase space shrinks to zero as t → ∞.

The Stable Manifold Theorem [22] asserts if g is a C^1 diffeomorphism with an equilibrium point **0** ε $\Re^m$ (or an equilibrium point translated to the origin) and a Jacobian matrix J**g** (**0**) with h distinct eigenvalues with negative real parts and (m – h) distinct eigenvalues with positive real parts, then **g** is hyperbolic attractor and **0** ε $\Re^m$ is a hyperbolic fixed-point. The local stable and unstable manifolds S and U of class C^1 are tangent to the stable and unstable subspaces (E^s) and (E^u), respectively, of the same dim such that:

$\forall$z ε S, m ≥ 0, $\mathbf{g}^m$ (S) ε S and $\mathbf{g}^\mathbf{m}$ (z) → **0** as m → ∞; and

$\forall$ z ε U, m ≥ 0, $\mathbf{g}^m$ (.) ε U and $\mathbf{g}^m$ (.) → 0 as m → ∞.

It then follows that the global stable and unstable manifolds are:

M^s (**0**) = $\cup_{m \geq 0}$ $\mathbf{g}^m$**(S)** and M^u **(0) =** $\cup_{m \geq 0}$ $\mathbf{g}^m$ (U), and M^s and M^u are invariant manifolds of class C^1. It can briefly be stated that the route to chaos in hyperbolic attractors starts with a transversal intersection between M^s (.) and M^u (.) where stable and unstable orbits cross at some other point z*. As both manifolds are invariant under **g**, all iterates of z***** lie at the intersection of $M^\mathbf{s}$ and $M^\mathbf{u}$. If so, it is then certain that a huge number of homoclinic transverse points will accumulate at **0** ε $\Re^m$, leading finally to a ‘homoclinic tangle’. In a homoclinic tangle hiher iterates of **g** produce the famous Smale’s [22] horseshoe which has a ‘strange’ invariant Cantor point set. The homoclinic tangle has been associated with chaotic dynamics since the discovery of the occurrence of transverse homoclnic points by Henri Poincaré.

It has been argued that in hyperbolic attractors M^s (.) and M^u (.) intersect transversally at a positive angle and there is no continuous splitting between E^s and E^u along the chaotic invariant set B; the latter condition is due to the existence of periodic orbits embedded in B with a different number of unstable directions. For a more complete discussion of that phenomenon the reader is referred to Anishenko *et al*., [23], Kantz *et al*.[24], Kubo *et al*.[15], among others.

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